

Exercise 11.1

Question 1:

If a line makes angles 90° , 135° , 45° with x , y and z -axes respectively, find its direction cosines.

Answer

Let direction cosines of the line be l , m , and n .

$$l = \cos 90^\circ = 0$$

$$m = \cos 135^\circ = -\frac{1}{\sqrt{2}}$$

$$n = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

Therefore, the direction cosines of the line are $0, -\frac{1}{\sqrt{2}},$ and $\frac{1}{\sqrt{2}}.$

Question 2:

Find the direction cosines of a line which makes equal angles with the coordinate axes.

Answer

Let the direction cosines of the line make an angle α with each of the coordinate axes.

$$\therefore l = \cos \alpha, m = \cos \alpha, n = \cos \alpha$$

$$l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha = 1$$

$$\Rightarrow 3 \cos^2 \alpha = 1$$

$$\Rightarrow \cos^2 \alpha = \frac{1}{3}$$

$$\Rightarrow \cos \alpha = \pm \frac{1}{\sqrt{3}}$$

Thus, the direction cosines of the line, which is equally inclined to the coordinate axes,

are $\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}},$ and $\pm \frac{1}{\sqrt{3}}.$

Question 3:

If a line has the direction ratios $-18, 12, -4$, then what are its direction cosines?

Answer

If a line has direction ratios of $-18, 12,$ and -4 , then its direction cosines are

$$\frac{-18}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}, \frac{12}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}, \frac{-4}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}$$

$$\text{i.e., } \frac{-18}{22}, \frac{12}{22}, \frac{-4}{22}$$

$$\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$$

Thus, the direction cosines are $-\frac{9}{11}, \frac{6}{11},$ and $-\frac{2}{11}.$

Question 4:

Show that the points $(2, 3, 4), (-1, -2, 1), (5, 8, 7)$ are collinear.

Answer

The given points are A (2, 3, 4), B (-1, -2, 1), and C (5, 8, 7).

It is known that the direction ratios of line joining the points, (x_1, y_1, z_1) and (x_2, y_2, z_2) , are given by, $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$.

The direction ratios of AB are $(-1 - 2)$, $(-2 - 3)$, and $(1 - 4)$ i.e., -3, -5, and -3.

The direction ratios of BC are $(5 - (-1))$, $(8 - (-2))$, and $(7 - 1)$ i.e., 6, 10, and 6.

It can be seen that the direction ratios of BC are -2 times that of AB i.e., they are proportional.

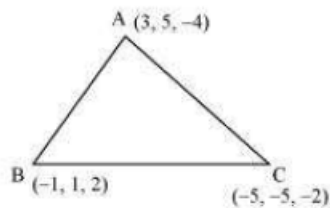
Therefore, AB is parallel to BC. Since point B is common to both AB and BC, points A, B, and C are collinear.

Question 5:

Find the direction cosines of the sides of the triangle whose vertices are (3, 5, -4), (-1, 1, 2) and (-5, -5, -2)

Answer

The vertices of $\triangle ABC$ are A (3, 5, -4), B (-1, 1, 2), and C (-5, -5, -2).



The direction ratios of side AB are $(-1 - 3)$, $(1 - 5)$, and $(2 - (-4))$ i.e., -4, -4, and 6.

$$\begin{aligned}\text{Then, } \sqrt{(-4)^2 + (-4)^2 + (6)^2} &= \sqrt{16 + 16 + 36} \\ &= \sqrt{68} \\ &= 2\sqrt{17}\end{aligned}$$

Therefore, the direction cosines of AB are

$$\begin{aligned}&\frac{-4}{\sqrt{(-4)^2 + (-4)^2 + (6)^2}}, \frac{-4}{\sqrt{(-4)^2 + (-4)^2 + (6)^2}}, \frac{6}{\sqrt{(-4)^2 + (-4)^2 + (6)^2}} \\&\frac{-4}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}, \frac{6}{2\sqrt{17}} \\&\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\end{aligned}$$

The direction ratios of BC are $(-5 - (-1))$, $(-5 - 1)$, and $(-2 - 2)$ i.e., -4, -6, and -4.

Therefore, the direction cosines of BC are

$$\begin{aligned}&\frac{-4}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}}, \frac{-6}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}}, \frac{-4}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}} \\&\frac{-4}{2\sqrt{17}}, \frac{-6}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}\end{aligned}$$

The direction ratios of CA are $(-5 - 3)$, $(-5 - 5)$, and $(-2 - (-4))$ i.e., -8, -10, and 2.

Therefore, the direction cosines of AC are

$$\begin{aligned}&\frac{-8}{\sqrt{(-8)^2 + (10)^2 + (2)^2}}, \frac{-10}{\sqrt{(-8)^2 + (10)^2 + (2)^2}}, \frac{2}{\sqrt{(-8)^2 + (10)^2 + (2)^2}} \\&\frac{-8}{2\sqrt{42}}, \frac{-10}{2\sqrt{42}}, \frac{2}{2\sqrt{42}}\end{aligned}$$

Exercise 11.2

Question 1:

Show that the three lines with direction cosines

$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}, \frac{4}{13}, \frac{12}{13}, \frac{3}{13}, \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ are mutually perpendicular.

Answer

Two lines with direction cosines, l_1, m_1, n_1 and l_2, m_2, n_2 , are perpendicular to each other, if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

(i) For the lines with direction cosines, $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ and $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$, we obtain

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \frac{12}{13} \times \frac{4}{13} + \left(\frac{-3}{13}\right) \times \frac{12}{13} + \left(\frac{-4}{13}\right) \times \frac{3}{13} \\ &= \frac{48}{169} - \frac{36}{169} - \frac{12}{169} \\ &= 0 \end{aligned}$$

Therefore, the lines are perpendicular.

(ii) For the lines with direction cosines, $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ and $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$, we obtain

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \frac{4}{13} \times \frac{3}{13} + \frac{12}{13} \times \left(\frac{-4}{13}\right) + \frac{3}{13} \times \frac{12}{13} \\ &= \frac{12}{169} - \frac{48}{169} + \frac{36}{169} \\ &= 0 \end{aligned}$$

Therefore, the lines are perpendicular.

(iii) For the lines with direction cosines, $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ and $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$, we obtain

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \left(\frac{3}{13}\right) \times \left(\frac{12}{13}\right) + \left(\frac{-4}{13}\right) \times \left(\frac{-3}{13}\right) + \left(\frac{12}{13}\right) \times \left(\frac{-4}{13}\right) \\ &= \frac{36}{169} + \frac{12}{169} - \frac{48}{169} \\ &= 0 \end{aligned}$$

Therefore, the lines are perpendicular.

Thus, all the lines are mutually perpendicular.

Question 2:

Show that the line through the points (1, -1, 2) (3, 4, -2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

Answer

Let AB be the line joining the points, (1, -1, 2) and (3, 4, -2), and CD be the line joining the points, (0, 3, 2) and (3, 5, 6).

The direction ratios, a_1, b_1, c_1 , of AB are (3 - 1), (4 - (-1)), and (-2 - 2) i.e., 2, 5, and -4.

The direction ratios, a_2, b_2, c_2 , of CD are (3 - 0), (5 - 3), and (6 - 2) i.e., 3, 2, and 4.

AB and CD will be perpendicular to each other, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$$\begin{aligned} a_1a_2 + b_1b_2 + c_1c_2 &= 2 \times 3 + 5 \times 2 + (-4) \times 4 \\ &= 6 + 10 - 16 \\ &= 0 \end{aligned}$$

Therefore, AB and CD are perpendicular to each other.

Question 3:

Show that the line through the points (4, 7, 8) (2, 3, 4) is parallel to the line through the points (-1, -2, 1), (1, 2, 5).

Answer

Let AB be the line through the points, (4, 7, 8) and (2, 3, 4), and CD be the line through the points, (-1, -2, 1) and (1, 2, 5).

The directions ratios, a_1, b_1, c_1 , of AB are (2 - 4), (3 - 7), and (4 - 8) i.e., -2, -4, and -4.

The direction ratios, a_2, b_2, c_2 , of CD are (1 - (-1)), (2 - (-2)), and (5 - 1) i.e., 2, 4, and 4.

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

AB will be parallel to CD, if

$$\frac{a_1}{a_2} = \frac{-2}{2} = -1$$

$$\frac{b_1}{b_2} = \frac{-4}{4} = -1$$

$$\frac{c_1}{c_2} = \frac{-4}{4} = -1$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, AB is parallel to CD.

Question 4:

Find the equation of the line which passes through the point (1, 2, 3) and is parallel to

the vector $3\hat{i} + 2\hat{j} - 2\hat{k}$.

Answer

It is given that the line passes through the point A (1, 2, 3). Therefore, the position

vector through A is $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$$

It is known that the line which passes through point A and parallel to \vec{b} is given by

$\vec{r} = \vec{a} + \lambda \vec{b}$, where λ is a constant.

$$\Rightarrow \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$$

This is the required equation of the line.

Question 5:

Find the equation of the line in vector and in Cartesian form that passes through the

point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction $\hat{i} + 2\hat{j} - \hat{k}$.

Answer

It is given that the line passes through the point with position vector

$$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k} \quad \dots(1)$$

$$\vec{b} = \hat{i} + 2\hat{j} - \hat{k} \quad \dots(2)$$

It is known that a line through a point with position vector \vec{a} and parallel to \vec{b} is given by

the equation, $\vec{r} = \vec{a} + \lambda \vec{b}$

$$\Rightarrow \vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + 2\hat{j} - \hat{k})$$

This is the required equation of the line in vector form.

$$\vec{r} = x\hat{i} - y\hat{j} + z\hat{k}$$

$$\Rightarrow x\hat{i} - y\hat{j} + z\hat{k} = (\lambda + 2)\hat{i} + (2\lambda - 1)\hat{j} + (-\lambda + 4)\hat{k}$$

Eliminating λ , we obtain the Cartesian form equation as

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

This is the required equation of the given line in Cartesian form.

Question 6:

Find the Cartesian equation of the line which passes through the point

$$(-2, 4, -5) \text{ and parallel to the line given by } \frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$$

Answer

It is given that the line passes through the point $(-2, 4, -5)$ and is parallel to

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$$

The direction ratios of the line, $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$, are 3, 5, and 6.

The required line is parallel to $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Therefore, its direction ratios are $3k$, $5k$, and $6k$, where $k \neq 0$

It is known that the equation of the line through the point (x_1, y_1, z_1) and with direction

$$\text{ratios, } a, b, c, \text{ is given by } \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

Therefore the equation of the required line is

$$\begin{aligned} \frac{x+2}{3k} &= \frac{y-4}{5k} = \frac{z+5}{6k} \\ \Rightarrow \frac{x+2}{3} &= \frac{y-4}{5} = \frac{z+5}{6} = k \end{aligned}$$

Question 7:

The Cartesian equation of a line is $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. Write its vector form.

Answer

The Cartesian equation of the line is

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2} \quad \dots (1)$$

The given line passes through the point $(5, -4, 6)$. The position vector of this point is

$$\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}$$

Also, the direction ratios of the given line are 3, 7, and 2.

This means that the line is in the direction of vector, $\vec{b} = 3\hat{i} + 7\hat{j} + 2\hat{k}$

It is known that the line through position vector \vec{a} and in the direction of the vector \vec{b} is

given by the equation, $\vec{r} = \vec{a} + \lambda\vec{b}, \lambda \in R$

$$\Rightarrow \vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$$

This is the required equation of the given line in vector form.

Question 8:

Find the vector and the Cartesian equations of the lines that pass through the origin and $(5, -2, 3)$.

Answer

The required line passes through the origin. Therefore, its position vector is given by,

$$\vec{a} = \vec{0} \quad \dots (1)$$

The direction ratios of the line through origin and $(5, -2, 3)$ are

$$(5 - 0) = 5, (-2 - 0) = -2, (3 - 0) = 3$$

The line is parallel to the vector given by the equation, $\vec{b} = 5\hat{i} - 2\hat{j} + 3\hat{k}$

The equation of the line in vector form through a point with position vector \vec{a} and parallel

to \vec{b} is, $\vec{r} = \vec{a} + \lambda\vec{b}, \lambda \in R$

$$\Rightarrow \vec{r} = \vec{0} + \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$$

$$\Rightarrow \vec{r} = \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$$

The equation of the line through the point (x_1, y_1, z_1) and direction ratios a, b, c is given

$$\text{by, } \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

Therefore, the equation of the required line in the Cartesian form is

$$\frac{x-0}{5} = \frac{y-0}{-2} = \frac{z-0}{3}$$

$$\Rightarrow \frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$$

Question 9:

Find the vector and the Cartesian equations of the line that passes through the points $(3, -2, -5), (3, -2, 6)$.

Answer

Let the line passing through the points, P $(3, -2, -5)$ and Q $(3, -2, 6)$, be PQ.

Since PQ passes through P $(3, -2, -5)$, its position vector is given by,

$$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$$

The direction ratios of PQ are given by,

$$(3 - 3) = 0, (-2 + 2) = 0, (6 + 5) = 11$$

The equation of the vector in the direction of PQ is

$$\vec{b} = 0\hat{i} - 0\hat{j} + 11\hat{k} = 11\hat{k}$$

The equation of PQ in vector form is given by, $\vec{r} = \vec{a} + \lambda\vec{b}, \lambda \in R$

$$\Rightarrow \vec{r} = (3\hat{i} - 2\hat{j} - 5\hat{k}) + 11\lambda\hat{k}$$

The equation of PQ in Cartesian form is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \quad \text{i.e.,}$$

$$\frac{x-3}{0} = \frac{y+2}{0} = \frac{z+5}{11}$$

Question 10:

Find the angle between the following pairs of lines:

$$\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k}) \text{ and}$$

(i)

$$\vec{r} = 7\hat{i} - 6\hat{j} + 4\hat{k} + \mu(2\hat{i} + 3\hat{j} + 2\hat{k})$$

$$r = 11 - 6k + \mu(i + 2j + 2k)$$

$$(ii) \quad \vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k}) \text{ and}$$

$$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$$

Answer

(i) Let Q be the angle between the given lines.

$$\cos Q = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

The angle between the given pairs of lines is given by,

The given lines are parallel to the vectors, $\vec{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}$ and $\vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$, respectively.

$$\therefore |\vec{b}_1| = \sqrt{3^2 + 2^2 + 6^2} = 7$$

$$|\vec{b}_2| = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$$

$$\begin{aligned} \vec{b}_1 \cdot \vec{b}_2 &= (3\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k}) \\ &= 3 \times 1 + 2 \times 2 + 6 \times 2 \\ &= 3 + 4 + 12 \\ &= 19 \end{aligned}$$

$$\Rightarrow \cos Q = \frac{19}{7 \times 3}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{19}{21}\right)$$

(ii) The given lines are parallel to the vectors, $\vec{b}_1 = \hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}$, respectively.

$$\therefore |\vec{b}_1| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$$

$$|\vec{b}_2| = \sqrt{(3)^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$$

$$\begin{aligned} \vec{b}_1 \cdot \vec{b}_2 &= (\hat{i} - \hat{j} - 2\hat{k}) \cdot (3\hat{i} - 5\hat{j} - 4\hat{k}) \\ &= 1 \cdot 3 - 1(-5) - 2(-4) \\ &= 3 + 5 + 8 \\ &= 16 \end{aligned}$$

$$\cos Q = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

$$\Rightarrow \cos Q = \frac{16}{\sqrt{6} \cdot 5\sqrt{2}} = \frac{16}{\sqrt{2} \cdot \sqrt{3} \cdot 5\sqrt{2}} = \frac{16}{10\sqrt{3}}$$

$$\Rightarrow \cos Q = \frac{8}{5\sqrt{3}}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{8}{5\sqrt{3}}\right)$$

Question 11:

Find the angle between the following pairs of lines:

$$(i) \quad \frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$$

$$(ii) \quad \frac{x}{2} = \frac{y}{2} = \frac{z}{1} \text{ and } \frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$$

i. Answer

ii. Let \vec{b}_1 and \vec{b}_2 be the vectors parallel to the pair of lines,

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}, \text{ respectively.}$$

$$\therefore \vec{b}_1 = 2\hat{i} + 5\hat{j} - 3\hat{k} \text{ and } \vec{b}_2 = -\hat{i} + 8\hat{j} + 4\hat{k}$$

$$|\vec{b}_1| = \sqrt{(2)^2 + (5)^2 + (-3)^2} = \sqrt{38}$$

$$|\vec{b}_2| = \sqrt{(-1)^2 + (8)^2 + (4)^2} = \sqrt{81} = 9$$

$$\begin{aligned} \vec{b}_1 \cdot \vec{b}_2 &= (2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (-\hat{i} + 8\hat{j} + 4\hat{k}) \\ &= 2(-1) + 5 \times 8 + (-3) \cdot 4 \\ &= -2 + 40 - 12 \\ &= 26 \end{aligned}$$

The angle, Q , between the given pair of lines is given by the relation,

$$\cos Q = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

$$\Rightarrow \cos Q = \frac{26}{9\sqrt{38}}$$

$$\Rightarrow Q = \cos^{-1} \left(\frac{26}{9\sqrt{38}} \right)$$

(ii) Let \vec{b}_1, \vec{b}_2 be the vectors parallel to the given pair of lines, $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$ and

$$\frac{x-5}{4} = \frac{y-5}{1} = \frac{z-3}{8}, \text{ respectively.}$$

$$\vec{b}_1 = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{b}_2 = 4\hat{i} + \hat{j} + 8\hat{k}$$

$$\therefore |\vec{b}_1| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{9} = 3$$

$$|\vec{b}_2| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$$

$$\begin{aligned} \vec{b}_1 \cdot \vec{b}_2 &= (2\hat{i} + 2\hat{j} + \hat{k}) \cdot (4\hat{i} + \hat{j} + 8\hat{k}) \\ &= 2 \times 4 + 2 \times 1 + 1 \times 8 \\ &= 8 + 2 + 8 \\ &= 18 \end{aligned}$$

$$\cos Q = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

If Q is the angle between the given pair of lines, then

$$\Rightarrow \cos Q = \frac{18}{3 \times 9} = \frac{2}{3}$$

$$\Rightarrow Q = \cos^{-1} \left(\frac{2}{3} \right)$$

Question 12:

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2} \text{ and}$$

Find the values of p so the line

$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5} \text{ are at right angles.}$$

Answer

The given equations can be written in the standard form as

$$\frac{x-1}{-3} = \frac{y-2}{\frac{2p}{7}} = \frac{z-3}{2} \quad \frac{x-1}{\frac{-3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$$

and

The direction ratios of the lines are $-3, \frac{2p}{7}, 2$ and $\frac{-3p}{7}, 1, -5$ respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 , are perpendicular to each other, if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\therefore (-3) \cdot \left(\frac{-3p}{7}\right) + \left(\frac{2p}{7}\right) \cdot (1) + 2 \cdot (-5) = 0$$

$$\Rightarrow \frac{9p}{7} + \frac{2p}{7} = 10$$

$$\Rightarrow 11p = 70$$

$$\Rightarrow p = \frac{70}{11}$$

Thus, the value of p is $\frac{70}{11}$.

Question 13:

Show that the lines $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ are perpendicular to each other.

Answer

The equations of the given lines are $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$ and $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

The direction ratios of the given lines are 7, -5, 1 and 1, 2, 3 respectively.

Two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 , are perpendicular to each other, if

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\therefore 7 \times 1 + (-5) \times 2 + 1 \times 3$$

$$= 7 - 10 + 3$$

$$= 0$$

Therefore, the given lines are perpendicular to each other.

Question 14:

Find the shortest distance between the lines

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \text{ and}$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

Answer

The equations of the given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \frac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|} \quad \dots(1)$$

Comparing the given equations, we obtain

$$\vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{b}_1 = \hat{i} - \hat{j} + \hat{k}$$

$$\vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k}$$

$$\vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = (2\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 2\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\vec{b}_1 \times \vec{b}_2 = (-2-1)\hat{i} - (2-2)\hat{j} + (1+2)\hat{k} = -3\hat{i} + 3\hat{k}$$

$$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-3)^2 + (3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

Substituting all the values in equation (1), we obtain

$$d = \left| \frac{(-3\hat{i} + 3\hat{k}) \cdot (\hat{i} - 3\hat{j} - 2\hat{k})}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \left| \frac{-3.1 + 3(-2)}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \left| \frac{-9}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}$$

$$\frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the two lines is $\frac{3\sqrt{2}}{2}$ units.

Question 15:

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \quad \text{and} \quad \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

Find the shortest distance between the lines

Answer

$$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1} \quad \text{and} \quad \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$$

The given lines are

It is known that the shortest distance between the two lines,

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \quad \text{and} \quad \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}, \text{ is given by,}$$

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \quad \dots(1)$$

Comparing the given equations, we obtain

$$x_1 = -1, \quad y_1 = -1, \quad z_1 = -1$$

$$a_1 = 7, \quad b_1 = -6, \quad c_1 = 1$$

$$x_2 = 3, \quad y_2 = 5, \quad z_2 = 7$$

$$a_2 = 1, \quad b_2 = -2, \quad c_2 = 1$$

$$\text{Then, } \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \end{vmatrix} = \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \end{vmatrix}$$

$$\begin{vmatrix} a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 1 \end{vmatrix} \\
 = 4(-6+2) - 6(7-1) + 8(-14+6) \\
 = -16 - 36 - 64 \\
 = -116$$

$$\begin{aligned}
 \Rightarrow \sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2} &= \sqrt{(-6+2)^2 + (1+7)^2 + (-14+6)^2} \\
 &= \sqrt{16+36+64} \\
 &= \sqrt{116} \\
 &= 2\sqrt{29}
 \end{aligned}$$

Substituting all the values in equation (1), we obtain

$$d = \frac{-116}{2\sqrt{29}} = \frac{-58}{\sqrt{29}} = \frac{-2 \times 29}{\sqrt{29}} = -2\sqrt{29}$$

Since distance is always non-negative, the distance between the given lines is

$$2\sqrt{29} \text{ units.}$$

Question 16:

Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$$

$$\text{and } \vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

Answer

The given lines are $\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$ and $\vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \frac{\left| (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) \right|}{\left| \vec{b}_1 \times \vec{b}_2 \right|} \quad \dots(1)$$

Comparing the given equations with $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, we obtain

$$\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$$

$$\vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$$

$$\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-3-6)\hat{i} - (1-4)\hat{j} + (3+6)\hat{k} = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\Rightarrow \left| \vec{b}_1 \times \vec{b}_2 \right| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$$

$$\begin{aligned}
 (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) &= (-9\hat{i} + 3\hat{j} + 9\hat{k}) \cdot (3\hat{i} + 3\hat{j} + 3\hat{k}) \\
 &= -9 \times 3 + 3 \times 3 + 9 \times 3 \\
 &= 9
 \end{aligned}$$

Substituting all the values in equation (1), we obtain

$$d = \frac{9}{3\sqrt{19}} = \frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the two given lines is $\frac{3}{\sqrt{19}}$ units.

Question 17:

Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \text{ and}$$

$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$

Answer

The given lines are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k}$$

$$\Rightarrow \vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}) \quad \dots(1)$$

$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$

$$\Rightarrow \vec{r} = (\hat{i} - \hat{j} + \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}) \quad \dots(2)$$

It is known that the shortest distance between the lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, is given by,

$$d = \frac{|\left(\vec{b}_1 \times \vec{b}_2\right) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|} \quad \dots(3)$$

For the given equations,

$$\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}$$

$$\vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$$

$$\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}$$

$$\vec{b}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \hat{j} - 4\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = (-2+4)\hat{i} - (2+2)\hat{j} + (-2-1)\hat{k} = 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(2)^2 + (-4)^2 + (-3)^2} = \sqrt{4+16+9} = \sqrt{29}$$

$$\therefore (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (2\hat{i} - 4\hat{j} - 3\hat{k}) \cdot (\hat{j} - 4\hat{k}) = -4 + 12 = 8$$

Substituting all the values in equation (3), we obtain

$$d = \frac{8}{\sqrt{29}} = \frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the lines is $\frac{8}{\sqrt{29}}$ units.

Exercise 11.3

Question 1:

In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a) $z = 2$ (b) $x + y + z = 1$

(c) $2x + 3y - z = 5$ (d) $5y + 8 = 0$

Answer

(a) The equation of the plane is $z = 2$ or $0x + 0y + z = 2 \dots (1)$

The direction ratios of normal are 0, 0, and 1.

$$\therefore \sqrt{0^2 + 0^2 + 1^2} = 1$$

Dividing both sides of equation (1) by 1, we obtain

$$0.x + 0.y + 1.z = 2$$

This is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of the perpendicular drawn from the origin.

Therefore, the direction cosines are 0, 0, and 1 and the distance of the plane from the origin is 2 units.

(b) $x + y + z = 1 \dots (1)$

The direction ratios of normal are 1, 1, and 1.

$$\therefore \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

Dividing both sides of equation (1) by $\sqrt{3}$, we obtain

$$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}} \dots (2)$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Therefore, the direction cosines of the normal are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$, and $\frac{1}{\sqrt{3}}$ and the distance of normal from the origin is $\frac{1}{\sqrt{3}}$ units.

(c) $2x + 3y - z = 5 \dots (1)$

The direction ratios of normal are 2, 3, and -1.

$$\therefore \sqrt{(2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$$

Dividing both sides of equation (1) by $\sqrt{14}$, we obtain

$$\frac{2}{\sqrt{14}}x + \frac{3}{\sqrt{14}}y - \frac{1}{\sqrt{14}}z = \frac{5}{\sqrt{14}}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Therefore, the direction cosines of the normal to the plane are $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$, and $\frac{-1}{\sqrt{14}}$ and

the distance of normal from the origin is $\frac{5}{\sqrt{14}}$ units.

(d) $5y + 8 = 0$

$$\Rightarrow 0x - 5y + 0z = 8 \dots (1)$$

The direction ratios of normal are 0, -5, and 0.

$$\therefore \sqrt{0+(-5)^2+0} = 5$$

Dividing both sides of equation (1) by 5, we obtain

$$-y = \frac{8}{5}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

Therefore, the direction cosines of the normal to the plane are 0, -1, and 0 and the

distance of normal from the origin is $\frac{8}{5}$ units.

Question 2:

Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector $3\hat{i} + 5\hat{j} - 6\hat{k}$.

Answer

The normal vector is, $\vec{n} = 3\hat{i} + 5\hat{j} - 6\hat{k}$

$$\therefore \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{(3)^2 + (5)^2 + (6)^2}} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}$$

It is known that the equation of the plane with position vector \vec{r} is given by, $\vec{r} \cdot \hat{n} = d$

$$\Rightarrow \vec{r} \cdot \left(\frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}} \right) = 7$$

This is the vector equation of the required plane.

Question 3:

Find the Cartesian equation of the following planes:

$$(a) \vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2 \quad (b) \vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$$

$$(c) \vec{r} \cdot [(s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}] = 15$$

Answer

(a) It is given that equation of the plane is

$$\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2 \quad \dots(1)$$

For any arbitrary point P (x, y, z) on the plane, position vector \vec{r} is given by,

$$\vec{r} = x\hat{i} + y\hat{j} - z\hat{k}$$

Substituting the value of \vec{r} in equation (1), we obtain

$$(x\hat{i} + y\hat{j} - z\hat{k}) \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$$

$$\Rightarrow x + y - z = 2$$

This is the Cartesian equation of the plane.

$$(b) \vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1 \quad \dots(1)$$

For any arbitrary point P (x, y, z) on the plane, position vector \vec{r} is given by,

$$\vec{r} = x\hat{i} + y\hat{j} - z\hat{k}$$

Substituting the value of \vec{r} in equation (1), we obtain

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$$

$$\Rightarrow 2x + 3y - 4z = 1$$

This is the Cartesian equation of the plane.

$$\text{(c)} \quad \vec{r} \cdot [(s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}] = 15 \quad \dots(1)$$

For any arbitrary point P (x, y, z) on the plane, position vector \vec{r} is given by,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Substituting the value of \vec{r} in equation (1), we obtain

$$\begin{aligned} (x\hat{i} + y\hat{j} + z\hat{k}) \cdot [(s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}] &= 15 \\ \Rightarrow (s-2t)x + (3-t)y + (2s+t)z &= 15 \end{aligned}$$

This is the Cartesian equation of the given plane.

Question 4:

In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

$$\text{(a)} \quad 2x + 3y + 4z - 12 = 0 \quad \text{(b)} \quad 3y + 4z - 6 = 0$$

$$\text{(c)} \quad x + y + z = 1 \quad \text{(d)} \quad 5y + 8 = 0$$

Answer

(a) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1) .

$$2x + 3y + 4z - 12 = 0$$

$$\Rightarrow 2x + 3y + 4z = 12 \quad \dots (1)$$

The direction ratios of normal are 2, 3, and 4.

$$\therefore \sqrt{(2)^2 + (3)^2 + (4)^2} = \sqrt{29}$$

Dividing both sides of equation (1) by $\sqrt{29}$, we obtain

$$\frac{2}{\sqrt{29}}x + \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{12}{\sqrt{29}}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

$$(ld, md, nd).$$

Therefore, the coordinates of the foot of the perpendicular are

$$\left(\frac{2}{\sqrt{29}} \cdot \frac{12}{\sqrt{29}}, \frac{3}{\sqrt{29}} \cdot \frac{12}{\sqrt{29}}, \frac{4}{\sqrt{29}} \cdot \frac{12}{\sqrt{29}} \right) \text{ i.e., } \left(\frac{24}{29}, \frac{36}{29}, \frac{48}{29} \right).$$

(b) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1) .

$$3y + 4z - 6 = 0$$

$$\Rightarrow 0x + 3y + 4z = 6 \quad \dots (1)$$

The direction ratios of the normal are 0, 3, and 4.

$$\therefore \sqrt{0^2 + 3^2 + 4^2} = 5$$

Dividing both sides of equation (1) by 5, we obtain

$$0x + \frac{3}{5}y + \frac{4}{5}z = \frac{6}{5}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of

normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

(ld, md, nd) .

Therefore, the coordinates of the foot of the perpendicular are

$$\left(0, \frac{3}{5}, \frac{6}{5}\right) \text{ i.e., } \left(0, \frac{18}{25}, \frac{24}{25}\right).$$

(c) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1) .

$$x + y + z = 1 \dots (1)$$

The direction ratios of the normal are 1, 1, and 1.

$$\therefore \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Dividing both sides of equation (1) by $\sqrt{3}$, we obtain

$$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

(ld, md, nd) .

Therefore, the coordinates of the foot of the perpendicular are

$$\left(\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\right) \text{ i.e., } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

(d) Let the coordinates of the foot of perpendicular P from the origin to the plane be (x_1, y_1, z_1) .

$$5y + 8 = 0$$

$$\Rightarrow 0x - 5y + 0z = 8 \dots (1)$$

The direction ratios of the normal are 0, -5, and 0.

$$\therefore \sqrt{0^2 + (-5)^2 + 0^2} = 5$$

Dividing both sides of equation (1) by 5, we obtain

$$-y = \frac{8}{5}$$

This equation is of the form $lx + my + nz = d$, where l, m, n are the direction cosines of normal to the plane and d is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by

(ld, md, nd) .

Therefore, the coordinates of the foot of the perpendicular are

$$\left(0, -1\left(\frac{8}{5}\right), 0\right) \text{ i.e., } \left(0, -\frac{8}{5}, 0\right).$$

Question 5:

Find the vector and Cartesian equation of the planes

(a) that passes through the point $(1, 0, -2)$ and the normal to the plane is $\hat{i} + \hat{j} - \hat{k}$.

(b) that passes through the point $(1, 4, 6)$ and the normal vector to the plane is

$$\hat{i} - 2\hat{j} + \hat{k}$$

Answer

(a) The position vector of point $(1, 0, -2)$ is $\vec{a} = \hat{i} - 2\hat{k}$

The normal vector \vec{N} perpendicular to the plane is $\vec{N} = \hat{i} + \hat{j} - \hat{k}$

The vector equation of the plane is given by, $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

$$\Rightarrow [\vec{r} - (\hat{i} - 2\hat{k})] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0 \quad \dots(1)$$

\vec{r} is the position vector of any point P (x, y, z) in the plane.

$$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Therefore, equation (1) becomes

$$\begin{aligned} & [(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - 2\hat{k})] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0 \\ \Rightarrow & [(x-1)\hat{i} + y\hat{j} + (z+2)\hat{k}] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0 \\ \Rightarrow & (x-1) + y - (z+2) = 0 \\ \Rightarrow & x + y - z - 3 = 0 \\ \Rightarrow & x + y - z = 3 \end{aligned}$$

This is the Cartesian equation of the required plane.

(b) The position vector of the point $(1, 4, 6)$ is $\vec{a} = \hat{i} + 4\hat{j} + 6\hat{k}$

The normal vector \vec{N} perpendicular to the plane is $\vec{N} = \hat{i} - 2\hat{j} + \hat{k}$

The vector equation of the plane is given by, $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

$$\Rightarrow [\vec{r} - (\hat{i} + 4\hat{j} + 6\hat{k})] \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \quad \dots(1)$$

\vec{r} is the position vector of any point P (x, y, z) in the plane.

$$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Therefore, equation (1) becomes

$$\begin{aligned} & [(x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} + 4\hat{j} + 6\hat{k})] \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \\ \Rightarrow & [(x-1)\hat{i} + (y-4)\hat{j} + (z-6)\hat{k}] \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \\ \Rightarrow & (x-1) - 2(y-4) + (z-6) = 0 \\ \Rightarrow & x - 2y + z + 1 = 0 \end{aligned}$$

This is the Cartesian equation of the required plane.

Question 6:

Find the equations of the planes that passes through three points.

(a) $(1, 1, -1)$, $(6, 4, -5)$, $(-4, -2, 3)$

(b) $(1, 1, 0)$, $(1, 2, 1)$, $(-2, 2, -1)$

Answer

(a) The given points are A $(1, 1, -1)$, B $(6, 4, -5)$, and C $(-4, -2, 3)$.

$$\begin{vmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{vmatrix} = (12 - 10) - (18 - 20) - (-12 + 16) \\ = 2 + 2 - 4 \\ = 0$$

Since A, B, C are collinear points, there will be infinite number of planes passing through the given points.

(b) The given points are A $(1, 1, 0)$, B $(1, 2, 1)$, and C $(-2, 2, -1)$

Q6: The given points are A (1, 1, 0), B (1, 2, 1), and C (-2, 2, -1).

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{vmatrix} = (-2-2)-(2+2) = -8 \neq 0$$

Therefore, a plane will pass through the points A, B, and C.

It is known that the equation of the plane through the points, (x_1, y_1, z_1) , (x_2, y_2, z_2) , and (x_3, y_3, z_3) , is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (-2)(x-1)-3(y-1)+3z = 0$$

$$\Rightarrow -2x-3y+3z+2+3 = 0$$

$$\Rightarrow -2x-3y+3z = -5$$

$$\Rightarrow 2x+3y-3z = 5$$

This is the Cartesian equation of the required plane.

Question 7:

Find the intercepts cut off by the plane $2x + y - z = 5$

Answer

$$2x + y - z = 5 \quad \dots(1)$$

Dividing both sides of equation (1) by 5, we obtain

$$\frac{2}{5}x + \frac{y}{5} - \frac{z}{5} = 1$$

$$\Rightarrow \frac{x}{\frac{5}{2}} + \frac{y}{5} + \frac{z}{-5} = 1 \quad \dots(2)$$

It is known that the equation of a plane in intercept form is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, where a , b , c are the intercepts cut off by the plane at x , y , and z axes respectively.

Therefore, for the given equation,

$$a = \frac{5}{2}, b = 5, \text{ and } c = -5$$

Thus, the intercepts cut off by the plane are $\frac{5}{2}$, 5, and -5.

Question 8:

Find the equation of the plane with intercept 3 on the y -axis and parallel to ZOX plane.

Answer

The equation of the plane ZOX is

$$y = 0$$

Any plane parallel to it is of the form, $y = a$

Since the y -intercept of the plane is 3,

$$\therefore a = 3$$

Thus, the equation of the required plane is $y = 3$

Question 9:

Find the equation of the plane through the intersection of the planes

$$3x - y + 2z - 4 = 0 \text{ and } x + y + z - 2 = 0 \text{ and the point } (2, 2, 1)$$

Answer

The equation of any plane through the intersection of the planes,

$$3x - y + 2z - 4 = 0 \text{ and } x + y + z - 2 = 0, \text{ is}$$

$$(3x - y + 2z - 4) + \alpha(x + y + z - 2) = 0, \text{ where } \alpha \in \mathbb{R} \quad \dots(1)$$

The plane passes through the point $(2, 2, 1)$. Therefore, this point will satisfy equation (1).

$$\therefore (3 \times 2 - 2 + 2 \times 1 - 4) + \alpha(2 + 2 + 1 - 2) = 0$$

$$\Rightarrow 2 + 3\alpha = 0$$

$$\Rightarrow \alpha = -\frac{2}{3}$$

Substituting $\alpha = -\frac{2}{3}$ in equation (1), we obtain

$$(3x - y + 2z - 4) - \frac{2}{3}(x + y + z - 2) = 0$$

$$\Rightarrow 3(3x - y + 2z - 4) - 2(x + y + z - 2) = 0$$

$$\Rightarrow (9x - 3y + 6z - 12) - 2(x + y + z - 2) = 0$$

$$\Rightarrow 7x - 5y + 4z - 8 = 0$$

This is the required equation of the plane.

Question 10:

Find the vector equation of the plane passing through the intersection of the planes

$$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7, \quad \vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9 \text{ and through the point } (2, 1, 3)$$

Answer

$$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7 \text{ and } \vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$$

The equations of the planes are

$$\Rightarrow \vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) - 7 = 0 \quad \dots(1)$$

$$\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) - 9 = 0 \quad \dots(2)$$

The equation of any plane through the intersection of the planes given in equations (1) and (2) is given by,

$$[\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) - 7] + \lambda [\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) - 9] = 0, \text{ where } \lambda \in \mathbb{R}$$

$$\vec{r} \cdot [(2\hat{i} + 2\hat{j} - 3\hat{k}) + \lambda(2\hat{i} + 5\hat{j} + 3\hat{k})] = 9\lambda + 7$$

$$\vec{r} \cdot [(2 + 2\lambda)\hat{i} + (2 + 5\lambda)\hat{j} + (3\lambda - 3)\hat{k}] = 9\lambda + 7 \quad \dots(3)$$

The plane passes through the point $(2, 1, 3)$. Therefore, its position vector is given by,

$$\vec{r} = 2\hat{i} + \hat{j} + 3\hat{k}$$

Substituting in equation (3), we obtain

$$(2\hat{i} + \hat{j} + 3\hat{k}) \cdot [(2 + 2\lambda)\hat{i} + (2 + 5\lambda)\hat{j} + (3\lambda - 3)\hat{k}] = 9\lambda + 7$$

$$\Rightarrow (2 + 2\lambda) + (2 + 5\lambda) + (3\lambda - 3) = 9\lambda + 7$$

$$\Rightarrow 18\lambda - 3 = 9\lambda + 7$$

$$\Rightarrow 9\lambda = 10$$

$$\Rightarrow \lambda = \frac{10}{9}$$

Substituting $\lambda = \frac{10}{9}$ in equation (3), we obtain

$$\vec{r} \cdot \left(\frac{38}{9}\hat{i} + \frac{68}{9}\hat{j} + \frac{3}{9}\hat{k} \right) = 17$$

$$\Rightarrow \vec{r} \cdot (38\hat{i} + 68\hat{j} + 3\hat{k}) = 153$$

This is the vector equation of the required plane.

Question 11:

Find the equation of the plane through the line of intersection of the planes

$x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$

Answer

The equation of the plane through the intersection of the planes, $x + y + z = 1$ and $2x + 3y + 4z = 5$, is

$$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow (2\lambda + 1)x + (3\lambda + 1)y + (4\lambda + 1)z - (5\lambda + 1) = 0 \quad \dots(1)$$

The direction ratios, a_1, b_1, c_1 , of this plane are $(2\lambda + 1)$, $(3\lambda + 1)$, and $(4\lambda + 1)$.

The plane in equation (1) is perpendicular to $x - y + z = 0$

Its direction ratios, a_2, b_2, c_2 , are 1, -1, and 1.

Since the planes are perpendicular,

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$\Rightarrow (2\lambda + 1) - (3\lambda + 1) + (4\lambda + 1) = 0$$

$$\Rightarrow 3\lambda + 1 = 0$$

$$\Rightarrow \lambda = -\frac{1}{3}$$

Substituting $\lambda = -\frac{1}{3}$ in equation (1), we obtain

$$\frac{1}{3}x - \frac{1}{3}z + \frac{2}{3} = 0$$

$$\Rightarrow x - z + 2 = 0$$

This is the required equation of the plane.

Question 12:

Find the angle between the planes whose vector equations are

$$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5 \quad \text{and} \quad \vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$$

Answer

The equations of the given planes are $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5$ and $\vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$

It is known that if \vec{n}_1 and \vec{n}_2 are normal to the planes, $\vec{r} \cdot \vec{n}_1 = d_1$ and $\vec{r} \cdot \vec{n}_2 = d_2$, then the angle between them, Q , is given by,

$$\cos Q = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|} \quad \dots(1)$$

Here, $\vec{n}_1 = 2\hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{n}_2 = 3\hat{i} - 3\hat{j} + 5\hat{k}$

$$\therefore \vec{n}_1 \cdot \vec{n}_2 = (2\hat{i} + 2\hat{j} - 3\hat{k}) \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 2 \cdot 3 + 2 \cdot (-3) + (-3) \cdot 5 = -15$$

$$|\vec{n}_1| = \sqrt{(2)^2 + (2)^2 + (-3)^2} = \sqrt{17}$$

$$|\vec{n}_2| = \sqrt{(3)^2 + (-3)^2 + (5)^2} = \sqrt{43}$$

Substituting the value of $\vec{n} \cdot \vec{n}_2$, $|\vec{n}_1|$ and $|\vec{n}_2|$ in equation (1), we obtain

$$\cos Q = \left| \frac{-15}{\sqrt{17} \cdot \sqrt{43}} \right|$$

$$\Rightarrow \cos Q = \frac{15}{\sqrt{731}}$$

$$\Rightarrow \cos Q^{-1} = \left(\frac{15}{\sqrt{731}} \right)$$

Question 13:

In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

(a) $7x + 5y + 6z + 30 = 0$ and $3x - y - 10z + 4 = 0$

(b) $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

(c) $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

(d) $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

(e) $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

Answer

The direction ratios of normal to the plane, $L_1 : a_1x + b_1y + c_1z = 0$, are a_1, b_1, c_1 and

$L_2 : a_2x + b_2y + c_2z = 0$ are a_2, b_2, c_2 .

$$L_1 \parallel L_2, \text{ if } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$L_1 \perp L_2, \text{ if } a_1a_2 + b_1b_2 + c_1c_2 = 0$$

The angle between L_1 and L_2 is given by,

$$Q = \cos^{-1} \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

(a) The equations of the planes are $7x + 5y + 6z + 30 = 0$ and

$$3x - y - 10z + 4 = 0$$

Here, $a_1 = 7, b_1 = 5, c_1 = 6$

$$a_2 = 3, b_2 = -1, c_2 = -10$$

$$a_1a_2 + b_1b_2 + c_1c_2 = 7 \times 3 + 5 \times (-1) + 6 \times (-10) = -44 \neq 0$$

Therefore, the given planes are not perpendicular.

$$\frac{a_1}{a_2} = \frac{7}{3}, \frac{b_1}{b_2} = \frac{5}{-1} = -5, \frac{c_1}{c_2} = \frac{6}{-10} = \frac{-3}{5}$$

$$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

It can be seen that,

Therefore, the given planes are not parallel.

The angle between them is given by,

$$\left| \frac{7 \times 3 + 5 \times (-1) + 6 \times (-10)}{\sqrt{7^2 + 5^2 + 6^2} \cdot \sqrt{3^2 + (-1)^2 + (-10)^2}} \right|$$

$$\begin{aligned}
 Q &= \cos^{-1} \left| \frac{1 \times 5 + 5 \times (-1) + 0 \times (-10)}{\sqrt{(7)^2 + (5)^2 + (6)^2} \times \sqrt{(3)^2 + (-1)^2 + (-10)^2}} \right| \\
 &= \cos^{-1} \left| \frac{21 - 5 - 60}{\sqrt{110} \times \sqrt{110}} \right| \\
 &= \cos^{-1} \frac{44}{110} \\
 &= \cos^{-1} \frac{2}{5}
 \end{aligned}$$

(b) The equations of the planes are $2x + y + 3z - 2 = 0$ and $x - 2y + 5 = 0$

Here, $a_1 = 2, b_1 = 1, c_1 = 3$ and $a_2 = 1, b_2 = -2, c_2 = 0$

$$\therefore a_1 a_2 + b_1 b_2 + c_1 c_2 = 2 \times 1 + 1 \times (-2) + 3 \times 0 = 0$$

Thus, the given planes are perpendicular to each other.

(c) The equations of the given planes are $2x - 2y + 4z + 5 = 0$ and $3x - 3y + 6z - 1 = 0$

Here, $a_1 = 2, b_1 = -2, c_1 = 4$ and

$$a_2 = 3, b_2 = -3, c_2 = 6 \quad a_1 a_2 + b_1 b_2 + c_1 c_2 = 2 \times 3 + (-2)(-3) + 4 \times 6 = 6 + 6 + 24 = 36 \neq 0$$

Thus, the given planes are not perpendicular to each other.

$$\frac{a_1}{a_2} = \frac{2}{3}, \frac{b_1}{b_2} = \frac{-2}{-3} = \frac{2}{3} \text{ and } \frac{c_1}{c_2} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, the given planes are parallel to each other.

(d) The equations of the planes are $2x - y + 3z - 1 = 0$ and $2x - y + 3z + 3 = 0$

Here, $a_1 = 2, b_1 = -1, c_1 = 3$ and $a_2 = 2, b_2 = -1, c_2 = 3$

$$\frac{a_1}{a_2} = \frac{2}{2} = 1, \frac{b_1}{b_2} = \frac{-1}{-1} = 1 \text{ and } \frac{c_1}{c_2} = \frac{3}{3} = 1$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, the given lines are parallel to each other.

(e) The equations of the given planes are $4x + 8y + z - 8 = 0$ and $y + z - 4 = 0$

Here, $a_1 = 4, b_1 = 8, c_1 = 1$ and $a_2 = 0, b_2 = 1, c_2 = 1$

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 4 \times 0 + 8 \times 1 + 1 \times 1 = 9 \neq 0$$

Therefore, the given lines are not perpendicular to each other.

$$\frac{a_1}{a_2} = \frac{4}{0}, \frac{b_1}{b_2} = \frac{8}{1} = 8, \frac{c_1}{c_2} = \frac{1}{1} = 1$$

$$\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

Therefore, the given lines are not parallel to each other.

The angle between the planes is given by,

$$Q = \cos^{-1} \left| \frac{4 \times 0 + 8 \times 1 + 1 \times 1}{\sqrt{4^2 + 8^2 + 1^2} \times \sqrt{0^2 + 1^2 + 1^2}} \right| = \cos^{-1} \left| \frac{9}{9 \times \sqrt{2}} \right| = \cos^{-1} \left(\frac{1}{\sqrt{2}} \right) = 45^\circ$$

Question 14:

In the following cases, find the distance of each of the given points from the corresponding given plane.

Point Plane

$$(a) (0, 0, 0) \quad 3x - 4y + 12z = 3$$

$$(b) (3, -2, 1) \quad 2x - y + 2z + 3 = 0$$

$$(c) (2, 3, -5) \quad x + 2y - 2z = 9$$

$$(d) (-6, 0, 0) \quad 2x - 3y + 6z - 2 = 0$$

Answer

It is known that the distance between a point, $p(x_1, y_1, z_1)$, and a plane, $Ax + By + Cz = D$, is given by,

$$d = \left| \frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}} \right| \quad \dots(1)$$

(a) The given point is $(0, 0, 0)$ and the plane is $3x - 4y + 12z = 3$

$$\therefore d = \left| \frac{3 \times 0 - 4 \times 0 + 12 \times 0 - 3}{\sqrt{(3)^2 + (-4)^2 + (12)^2}} \right| = \frac{3}{\sqrt{169}} = \frac{3}{13}$$

(b) The given point is $(3, -2, 1)$ and the plane is $2x - y + 2z + 3 = 0$

$$\therefore d = \left| \frac{2 \times 3 - (-2) + 2 \times 1 + 3}{\sqrt{(2)^2 + (-1)^2 + (2)^2}} \right| = \left| \frac{13}{3} \right| = \frac{13}{3}$$

(c) The given point is $(2, 3, -5)$ and the plane is $x + 2y - 2z = 9$

$$\therefore d = \left| \frac{2 + 2 \times 3 - 2(-5) - 9}{\sqrt{(1)^2 + (2)^2 + (-2)^2}} \right| = \frac{9}{3} = 3$$

(d) The given point is $(-6, 0, 0)$ and the plane is $2x - 3y + 6z - 2 = 0$

$$d = \left| \frac{2(-6) - 3 \times 0 + 6 \times 0 - 2}{\sqrt{(2)^2 + (-3)^2 + (6)^2}} \right| = \left| \frac{-14}{\sqrt{49}} \right| = \frac{14}{7} = 2$$

Miscellaneous Solutions

Question 1:

Show that the line joining the origin to the point $(2, 1, 1)$ is perpendicular to the line determined by the points $(3, 5, -1)$, $(4, 3, -1)$.

Answer

Let OA be the line joining the origin, O $(0, 0, 0)$, and the point, A $(2, 1, 1)$.

Also, let BC be the line joining the points, B $(3, 5, -1)$ and C $(4, 3, -1)$.

The direction ratios of OA are 2, 1, and 1 and of BC are $(4 - 3) = 1$, $(3 - 5) = -2$, and $(-1 - 1) = -2$

OA is perpendicular to BC, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 1 + 1 \times (-2) + 1 \times (-2) = 2 - 2 - 2 = -2 \neq 0$$

Thus, OA is not perpendicular to BC.

Question 2:

If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines,

show that the direction cosines of the line perpendicular to both of these are $m_1n_2 - m_2n_1$, $n_1l_2 - n_2l_1$, $l_1m_2 - l_2m_1$.

Answer

It is given that l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two mutually perpendicular lines. Therefore,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \quad \dots(1)$$

$$l_1^2 + m_1^2 + n_1^2 = 1 \quad \dots(2)$$

$$l_2^2 + m_2^2 + n_2^2 = 1 \quad \dots(3)$$

Let l, m, n be the direction cosines of the line which is perpendicular to the line with direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 .

$$\therefore ll_1 + mm_1 + nn_1 = 0$$

$$ll_2 + mm_2 + nn_2 = 0$$

$$\therefore \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1}$$

$$\Rightarrow \frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_1l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_1)^2}$$

$$\Rightarrow \frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_1l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_1)^2}$$

$$= \frac{l^2 + m^2 + n^2}{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2} \quad \dots(4)$$

l, m, n are the direction cosines of the line.

$$\therefore l^2 + m^2 + n^2 = 1 \quad \dots (5)$$

It is known that,

$$(l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1l_2 + m_1m_2 + n_1n_2)^2 \\ = (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$$

From (1), (2), and (3), we obtain

$$\Rightarrow 1.1 - 0 = (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$$

$$\therefore (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 1 \quad \dots(6)$$

Substituting the values from equations (5) and (6) in equation (4), we obtain

$$\frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_1l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_1)^2} = 1$$

$$\Rightarrow l = m_1n_2 - m_2n_1, m = n_1l_2 - n_2l_1, n = l_1m_2 - l_2m_1$$

Thus, the direction cosines of the required line are $m_1n_2 - m_2n_1$, $n_1l_2 - n_2l_1$, and $l_1m_2 - l_2m_1$.

Question 3:

Find the angle between the lines whose direction ratios are a, b, c and $b - c, c - a, a - b$.

Answer

The angle Q between the lines with direction cosines, a, b, c and $b - c, c - a,$

$a - b$, is given by,

$$\cos Q = \left| \frac{a(b-c) + b(c-a) + c(a-b)}{\sqrt{a^2 + b^2 + c^2} + \sqrt{(b-c)^2 + (c-a)^2 + (a-b)^2}} \right|$$

$$\Rightarrow \cos Q = 0$$

$$\Rightarrow Q = \cos^{-1} 0$$

$$\Rightarrow Q = 90^\circ$$

Thus, the angle between the lines is 90° .

Question 4:

Find the equation of a line parallel to x-axis and passing through the origin.

Answer

The line parallel to x-axis and passing through the origin is x-axis itself.

Let A be a point on x-axis. Therefore, the coordinates of A are given by $(a, 0, 0)$, where $a \in \mathbb{R}$.

Direction ratios of OA are $(a - 0) = a, 0, 0$

The equation of OA is given by,

$$\frac{x-0}{a} = \frac{y-0}{0} = \frac{z-0}{0}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{0} = a$$

Thus, the equation of line parallel to x-axis and passing through origin is

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$$

Question 5:

If the coordinates of the points A, B, C, D be $(1, 2, 3)$, $(4, 5, 7)$, $(-4, 3, -6)$ and $(2, 9, 2)$ respectively, then find the angle between the lines AB and CD.

Answer

The coordinates of A, B, C, and D are $(1, 2, 3)$, $(4, 5, 7)$, $(-4, 3, -6)$, and $(2, 9, 2)$ respectively.

The direction ratios of AB are $(4 - 1) = 3$, $(5 - 2) = 3$, and $(7 - 3) = 4$

The direction ratios of CD are $(2 - (-4)) = 6$, $(9 - 3) = 6$, and $(2 - (-6)) = 8$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{1}{2}$$

It can be seen that,

Therefore, AB is parallel to CD.

Thus, the angle between AB and CD is either 0° or 180° .

Question 6:

If the lines $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ are perpendicular, find the value of k .

Answer

The direction of ratios of the lines, $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$ and $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$, are -3 , $2k$, 2 and $3k$, 1 , -5 respectively.

It is known that two lines with direction ratios, a_1, b_1, c_1 and a_2, b_2, c_2 , are perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$$\therefore -3(3k) + 2k \times 1 + 2(-5) = 0$$

$$\Rightarrow -9k + 2k - 10 = 0$$

$$\Rightarrow 7k = -10$$

$$\Rightarrow k = \frac{-10}{7}$$

Therefore, for $k = -\frac{10}{7}$, the given lines are perpendicular to each other.

Question 7:

Find the vector equation of the plane passing through (1, 2, 3) and perpendicular to the

$$\text{plane } \vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$$

Answer

The position vector of the point (1, 2, 3) is $\vec{r}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$

The direction ratios of the normal to the plane, $\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$, are 1, 2, and -5

and the normal vector is $\vec{N} = \hat{i} + 2\hat{j} - 5\hat{k}$

The equation of a line passing through a point and perpendicular to the given plane is

given by, $\vec{l} = \vec{r} + \lambda \vec{N}$, $\lambda \in R$

$$\Rightarrow \vec{l} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 5\hat{k})$$

Question 8:

Find the equation of the plane passing through (a, b, c) and parallel to the plane

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$$

Answer

Any plane parallel to the plane, $\vec{r}_1 \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$, is of the form

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = \lambda \quad \dots(1)$$

The plane passes through the point (a, b, c). Therefore, the position vector \vec{r} of this

point is $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$

Therefore, equation (1) becomes

$$(a\hat{i} + b\hat{j} + c\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = \lambda$$

$$\Rightarrow a + b + c = \lambda$$

Substituting $\lambda = a + b + c$ in equation (1), we obtain

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c \quad \dots(2)$$

This is the vector equation of the required plane.

Substituting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in equation (2), we obtain

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c$$

$$\Rightarrow x + y + z = a + b + c$$

Question 9:

Find the shortest distance between lines $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$

$$\text{and } \vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$$

Answer

The given lines are

$$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k}) \quad \dots(1)$$

$$\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k}) \quad \dots(2)$$

It is known that the shortest distance between two lines, $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$, is given by

$$d = \frac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|} \quad \dots(3)$$

Comparing $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ to equations (1) and (2), we obtain

$$\vec{a}_1 = 6\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\vec{b}_1 = \hat{i} - 2\hat{j} + 2\hat{k}$$

$$\vec{a}_2 = -4\hat{i} - \hat{k}$$

$$\vec{b}_2 = 3\hat{i} - 2\hat{j} - 2\hat{k}$$

$$\Rightarrow \vec{a}_2 - \vec{a}_1 = (-4\hat{i} - \hat{k}) - (6\hat{i} + 2\hat{j} + 2\hat{k}) = -10\hat{i} - 2\hat{j} - 3\hat{k}$$

$$\Rightarrow \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = (4+4)\hat{i} - (-2-6)\hat{j} + (-2+6)\hat{k} = 8\hat{i} + 8\hat{j} + 4\hat{k}$$

$$\therefore |\vec{b}_1 \times \vec{b}_2| = \sqrt{(8)^2 + (8)^2 + (4)^2} = 12$$

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (8\hat{i} + 8\hat{j} + 4\hat{k}) \cdot (-10\hat{i} - 2\hat{j} - 3\hat{k}) = -80 - 16 - 12 = -108$$

Substituting all the values in equation (3), we obtain

$$d = \frac{|-108|}{12} = 9$$

Therefore, the shortest distance between the two given lines is 9 units.

Question 10:

Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the YZ-plane

Answer

It is known that the equation of the line passing through the points, (x_1, y_1, z_1) and (x_2, y_2, z_2) , is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

The line passing through the points, (5, 1, 6) and (3, 4, 1), is given by,

$$\begin{aligned} \frac{x-5}{3-5} &= \frac{y-1}{4-1} = \frac{z-6}{1-6} \\ \Rightarrow \frac{x-5}{-2} &= \frac{y-1}{3} = \frac{z-6}{-5} = k \text{ (say)} \\ \Rightarrow x &= 5-2k, y = 3k+1, z = 6-5k \end{aligned}$$

Any point on the line is of the form $(5-2k, 3k+1, 6-5k)$.

The equation of YZ-plane is $x = 0$

Since the line passes through YZ-plane,

$$5-2k = 0$$

$$\therefore k = \frac{5}{2}$$

$$\Rightarrow \kappa = \frac{-5}{3}$$

$$\Rightarrow 3k + 1 = 3 \times \frac{5}{2} + 1 = \frac{17}{2}$$

$$6 - 5k = 6 - 5 \times \frac{5}{2} = \frac{-13}{2}$$

Therefore, the required point is $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$.

Question 11:

Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the ZX - plane.

Answer

It is known that the equation of the line passing through the points, (x_1, y_1, z_1) and $(x_2,$

$y_2, z_2)$, is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

The line passing through the points, (5, 1, 6) and (3, 4, 1), is given by,

$$\frac{x-5}{3-5} = \frac{y-1}{4-1} = \frac{z-6}{1-6}$$

$$\Rightarrow \frac{x-5}{-2} = \frac{y-1}{3} = \frac{z-6}{-5} = k \text{ (say)}$$

$$\Rightarrow x = 5 - 2k, y = 3k + 1, z = 6 - 5k$$

Any point on the line is of the form $(5 - 2k, 3k + 1, 6 - 5k)$.

Since the line passes through ZX-plane,

$$3k + 1 = 0$$

$$\Rightarrow k = -\frac{1}{3}$$

$$\Rightarrow 5 - 2k = 5 - 2\left(-\frac{1}{3}\right) = \frac{17}{3}$$

$$6 - 5k = 6 - 5\left(-\frac{1}{3}\right) = \frac{23}{3}$$

Therefore, the required point is $\left(\frac{17}{3}, 0, \frac{23}{3}\right)$.

Question 12:

Find the coordinates of the point where the line through (3, -4, -5) and (2, -3, 1) crosses the plane $2x + y + z = 7$.

Answer

It is known that the equation of the line through the points, (x_1, y_1, z_1) and (x_2, y_2, z_2) , is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Since the line passes through the points, (3, -4, -5) and (2, -3, 1), its equation is given by,

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5}$$

$$\Rightarrow \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = k \text{ (say)}$$

$$\Rightarrow x = 3 - k, y = k - 4, z = 6k - 5$$

Therefore, any point on the line is of the form $(3 - k, k - 4, 6k - 5)$.

This point lies on the plane, $2x + y + z = 7$

$$2(3 - k) + (k - 4) + (6k - 5) = 7$$

$$\therefore 2(3-k) + (k-4) + (6k-5) = 7$$

$$\Rightarrow 5k-3=7$$

$$\Rightarrow k=2$$

Hence, the coordinates of the required point are $(3-2, 2-4, 6 \times 2-5)$ i.e., $(1, -2, 7)$.

Question 13:

Find the equation of the plane passing through the point $(-1, 3, 2)$ and perpendicular to each of the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

Answer

The equation of the plane passing through the point $(-1, 3, 2)$ is

$$a(x+1) + b(y-3) + c(z-2) = 0 \dots (1)$$

where, a, b, c are the direction ratios of normal to the plane.

It is known that two planes, $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, are perpendicular, if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Plane (1) is perpendicular to the plane, $x + 2y + 3z = 5$

$$\therefore a \cdot 1 + b \cdot 2 + c \cdot 3 = 0$$

$$\Rightarrow a + 2b + 3c = 0 \dots (2)$$

Also, plane (1) is perpendicular to the plane, $3x + 3y + z = 0$

$$\therefore a \cdot 3 + b \cdot 3 + c \cdot 1 = 0$$

$$\Rightarrow 3a + 3b + c = 0 \dots (3)$$

From equations (2) and (3), we obtain

$$\frac{a}{2 \times 1 - 3 \times 3} = \frac{b}{3 \times 3 - 1 \times 1} = \frac{c}{1 \times 3 - 2 \times 3}$$

$$\Rightarrow \frac{a}{-7} = \frac{b}{8} = \frac{c}{-3} = k \text{ (say)}$$

$$\Rightarrow a = -7k, b = 8k, c = -3k$$

Substituting the values of a, b , and c in equation (1), we obtain

$$-7k(x+1) + 8k(y-3) - 3k(z-2) = 0$$

$$\Rightarrow (-7x-7) + (8y-24) - 3z+6 = 0$$

$$\Rightarrow -7x + 8y - 3z - 25 = 0$$

$$\Rightarrow 7x - 8y + 3z + 25 = 0$$

This is the required equation of the plane.

Question 14:

If the points $(1, 1, p)$ and $(-3, 0, 1)$ be equidistant from the plane

$$\vec{r} \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0, \text{ then find the value of } p.$$

Answer

The position vector through the point $(1, 1, p)$ is $\vec{a}_1 = \hat{i} + \hat{j} + p\hat{k}$

Similarly, the position vector through the point $(-3, 0, 1)$ is

$$\vec{a}_2 = -4\hat{i} + \hat{k}$$

The equation of the given plane is $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$

It is known that the perpendicular distance between a point whose position vector is

$$D = \frac{|\vec{a} \cdot \vec{N} - d|}{|\vec{N}|}$$

\vec{a} and the plane, $\vec{r} \cdot \vec{N} = d$, is given by, $|\vec{N}|$

Here, $\vec{N} = 3\hat{i} + 4\hat{j} - 12\hat{k}$ and $d = -13$

Therefore, the distance between the point $(1, 1, p)$ and the given plane is

$$D_1 = \frac{\left| (1\hat{i} + 1\hat{j} + p\hat{k}) \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 \right|}{|3\hat{i} + 4\hat{j} - 12\hat{k}|}$$

$$\Rightarrow D_1 = \frac{|3 + 4 - 12p + 13|}{\sqrt{3^2 + 4^2 + (-12)^2}}$$

$$\Rightarrow D_1 = \frac{|20 - 12p|}{13} \quad \dots(1)$$

Similarly, the distance between the point $(-3, 0, 1)$ and the given plane is

$$D_2 = \frac{\left| (-3\hat{i} + 0\hat{j} + 1\hat{k}) \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 \right|}{|3\hat{i} + 4\hat{j} - 12\hat{k}|}$$

$$\Rightarrow D_2 = \frac{|-9 - 12 + 13|}{\sqrt{3^2 + 4^2 + (-12)^2}}$$

$$\Rightarrow D_2 = \frac{8}{13} \quad \dots(2)$$

It is given that the distance between the required plane and the points, $(1, 1, p)$ and $(-3, 0, 1)$, is equal.

$$\therefore D_1 = D_2$$

$$\Rightarrow \frac{|20 - 12p|}{13} = \frac{8}{13}$$

$$\Rightarrow 20 - 12p = 8 \text{ or } -(20 - 12p) = 8$$

$$\Rightarrow 12p = 12 \text{ or } 12p = 28$$

$$\Rightarrow p = 1 \text{ or } p = \frac{7}{3}$$

Question 15:

Find the equation of the plane passing through the line of intersection of the planes

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1 \quad \text{and} \quad \vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0 \quad \text{and parallel to x-axis.}$$

Answer

The given planes are

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$$

$$\Rightarrow \vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) - 1 = 0$$

$$\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$$

The equation of any plane passing through the line of intersection of these planes is

$$\left[\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) - 1 \right] + \lambda \left[\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 \right] = 0$$

$$\vec{r} \cdot \left[(2\lambda + 1)\hat{i} + (3\lambda + 1)\hat{j} + (1 - \lambda)\hat{k} \right] + (4\lambda + 1) = 0 \quad \dots(1)$$

Its direction ratios are $(2\lambda + 1)$, $(3\lambda + 1)$, and $(1 - \lambda)$.

The required plane is parallel to x-axis. Therefore, its normal is perpendicular to x-axis.

The direction ratios of x-axis are 1, 0, and 0.

$$\therefore 1 \cdot (2\lambda + 1) + 0(3\lambda + 1) + 0(1 - \lambda) = 0$$

$$\Rightarrow 2\lambda + 1 = 0$$

$$\Rightarrow \lambda = -\frac{1}{2}$$

Substituting $\lambda = -\frac{1}{2}$ in equation (1), we obtain

$$\Rightarrow \vec{r} \cdot \left[-\frac{1}{2}\hat{j} + \frac{3}{2}\hat{k} \right] + (-3) = 0$$

$$\Rightarrow \vec{r} \cdot (\hat{j} - 3\hat{k}) + 6 = 0$$

Therefore, its Cartesian equation is $y - 3z + 6 = 0$

This is the equation of the required plane.

Question 16:

If O be the origin and the coordinates of P be (1, 2, -3), then find the equation of the plane passing through P and perpendicular to OP.

Answer

The coordinates of the points, O and P, are (0, 0, 0) and (1, 2, -3) respectively.

Therefore, the direction ratios of OP are $(1 - 0) = 1$, $(2 - 0) = 2$, and $(-3 - 0) = -3$

It is known that the equation of the plane passing through the point (x_1, y_1, z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \text{ where, } a, b, \text{ and } c \text{ are the direction ratios of normal.}$$

Here, the direction ratios of normal are 1, 2, and -3 and the point P is (1, 2, -3).

Thus, the equation of the required plane is

$$1(x - 1) + 2(y - 2) - 3(z + 3) = 0$$

$$\Rightarrow x + 2y - 3z - 14 = 0$$

Question 17:

Find the equation of the plane which contains the line of intersection of the planes

$$\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0, \quad \vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0 \text{ and which is perpendicular to the plane}$$

$$\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$$

Answer

The equations of the given planes are

$$\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0 \quad \dots(1)$$

$$\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0 \quad \dots(2)$$

The equation of the plane passing through the line intersection of the plane given in equation (1) and equation (2) is

$$[\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4] + \lambda [\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5] = 0$$

$$\vec{r} \cdot [(2\lambda + 1)\hat{i} + (\lambda + 2)\hat{j} + (3 - \lambda)\hat{k}] + (5\lambda - 4) = 0 \quad \dots(3)$$

The plane in equation (3) is perpendicular to the plane, $\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$

$$\therefore 5(2\lambda + 1) + 3(\lambda + 2) - 6(3 - \lambda) = 0$$

$$\Rightarrow 19\lambda - 7 = 0$$

$$\Rightarrow \lambda = \frac{7}{19}$$

Substituting $\lambda = \frac{1}{19}$ in equation (3), we obtain

$$\begin{aligned} \Rightarrow \vec{r} \cdot \left[\frac{33}{19}\hat{i} + \frac{45}{19}\hat{j} + \frac{50}{19}\hat{k} \right] \frac{-41}{19} &= 0 \\ \Rightarrow \vec{r} \cdot (33\hat{i} + 45\hat{j} + 50\hat{k}) - 41 &= 0 \end{aligned} \quad \dots(4)$$

This is the vector equation of the required plane.

The Cartesian equation of this plane can be obtained by substituting $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ in equation (3).

$$\begin{aligned} (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (33\hat{i} + 45\hat{j} + 50\hat{k}) - 41 &= 0 \\ \Rightarrow 33x + 45y + 50z - 41 &= 0 \end{aligned}$$

Question 18:

Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \quad \text{and the plane } \vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5.$$

Answer

The equation of the given line is

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \quad \dots(1)$$

The equation of the given plane is

$$\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5 \quad \dots(2)$$

Substituting the value of \vec{r} from equation (1) in equation (2), we obtain

$$\begin{aligned} [2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})] \cdot (\hat{i} - \hat{j} + \hat{k}) &= 5 \\ \Rightarrow [(3\lambda + 2)\hat{i} + (4\lambda - 1)\hat{j} + (2\lambda + 2)\hat{k}] \cdot (\hat{i} - \hat{j} + \hat{k}) &= 5 \\ \Rightarrow (3\lambda + 2) - (4\lambda - 1) + (2\lambda + 2) &= 5 \\ \Rightarrow \lambda &= 0 \end{aligned}$$

Substituting this value in equation (1), we obtain the equation of the line as

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k}$$

This means that the position vector of the point of intersection of the line and the plane

$$\text{is } \vec{r} = 2\hat{i} - \hat{j} + 2\hat{k}$$

This shows that the point of intersection of the given line and plane is given by the

coordinates, $(2, -1, 2)$. The point is $(-1, -5, -10)$.

The distance d between the points, $(2, -1, 2)$ and $(-1, -5, -10)$, is

$$d = \sqrt{(-1-2)^2 + (-5+1)^2 + (-10-2)^2} = \sqrt{9+16+144} = \sqrt{169} = 13$$

Question 19:

Find the vector equation of the line passing through $(1, 2, 3)$ and parallel to the planes

$$\vec{r} \cdot (\hat{i} - \hat{j} + 2\hat{k}) = 5 \quad \text{and} \quad \vec{r} \cdot (3\hat{i} + \hat{j} + \hat{k}) = 6.$$

Answer

Let the required line be parallel to vector \vec{b} given by,

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

The position vector of the point $(1, 2, 3)$ is $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

The equation of line passing through (1, 2, 3) and parallel to \vec{b} is given by,

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

$$\Rightarrow \vec{r}(\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \quad \dots(1)$$

The equations of the given planes are

$$\vec{r} \cdot (\hat{i} - \hat{j} + 2\hat{k}) = 5 \quad \dots(2)$$

$$\vec{r} \cdot (3\hat{i} + \hat{j} + \hat{k}) = 6 \quad \dots(3)$$

The line in equation (1) and plane in equation (2) are parallel. Therefore, the normal to the plane of equation (2) and the given line are perpendicular.

$$\Rightarrow (\hat{i} - \hat{j} + 2\hat{k}) \cdot \lambda(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = 0$$

$$\Rightarrow \lambda(b_1 - b_2 + 2b_3) = 0$$

$$\Rightarrow b_1 - b_2 + 2b_3 = 0 \quad \dots(4)$$

Similarly, $(3\hat{i} + \hat{j} + \hat{k}) \cdot \lambda(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = 0$

$$\Rightarrow \lambda(3b_1 + b_2 + b_3) = 0$$

$$\Rightarrow 3b_1 + b_2 + b_3 = 0 \quad \dots(5)$$

From equations (4) and (5), we obtain

$$\frac{b_1}{(-1) \times 1 - 1 \times 2} = \frac{b_2}{2 \times 3 - 1 \times 1} = \frac{b_3}{1 \times 1 - 3(-1)}$$

$$\Rightarrow \frac{b_1}{-3} = \frac{b_2}{5} = \frac{b_3}{4}$$

Therefore, the direction ratios of \vec{b} are -3, 5, and 4.

$$\therefore \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = -3\hat{i} + 5\hat{j} + 4\hat{k}$$

Substituting the value of \vec{b} in equation (1), we obtain

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(-3\hat{i} + 5\hat{j} + 4\hat{k})$$

This is the equation of the required line.

Question 20:

Find the vector equation of the line passing through the point (1, 2, -4) and

perpendicular to the two lines: $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

Answer

Let the required line be parallel to the vector \vec{b} given by, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

The position vector of the point (1, 2, -4) is $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$

The equation of the line passing through (1, 2, -4) and parallel to vector \vec{b} is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

$$\Rightarrow \vec{r}(\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \quad \dots(1)$$

The equations of the lines are

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \quad \dots(2)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad \dots(3)$$

Line (1) and line (2) are perpendicular to each other.

$$(-3\hat{i} + 5\hat{j} + 4\hat{k}) \cdot (3\hat{i} + \hat{j} + \hat{k}) = 0 \quad \dots(4)$$

$$\therefore 5b_1 - 10b_2 + 10b_3 = 0 \quad \dots(4)$$

Also, line (1) and line (3) are perpendicular to each other.

$$\therefore 3b_1 + 8b_2 - 5b_3 = 0 \quad \dots(5)$$

From equations (4) and (5), we obtain

$$\begin{aligned} \frac{b_1}{(-16)(-5) - 8 \times 7} &= \frac{b_2}{7 \times 3 - 3(-5)} = \frac{b_3}{3 \times 8 - 3(-16)} \\ \Rightarrow \frac{b_1}{24} &= \frac{b_2}{36} = \frac{b_3}{72} \\ \Rightarrow \frac{b_1}{2} &= \frac{b_2}{3} = \frac{b_3}{6} \end{aligned}$$

\therefore Direction ratios of \vec{b} are 2, 3, and 6.

$$\therefore \vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Substituting $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ in equation (1), we obtain

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

This is the equation of the required line.

Question 21:

Prove that if a plane has the intercepts a , b , c and is at a distance of P units from the

origin, then $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$

Answer

The equation of a plane having intercepts a , b , c with x , y , and z axes respectively is given by,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \dots(1)$$

The distance (p) of the plane from the origin is given by,

$$\begin{aligned} p &= \left| \frac{\frac{0}{a} + \frac{0}{b} + \frac{0}{c} - 1}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2}} \right| \\ \Rightarrow p &= \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} \\ \Rightarrow p^2 &= \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}} \\ \Rightarrow \frac{1}{p^2} &= \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \end{aligned}$$

Question 22:

Distance between the two planes: $2x + 3y + 4z = 4$ and $4x + 6y + 8z = 12$ is

(A) 2 units (B) 4 units (C) 8 units

(D) $\frac{2}{\sqrt{29}}$ units

Answer

Answer

The equations of the planes are

$$2x + 3y + 4z = 4 \quad \dots(1)$$

$$4x + 6y + 8z = 12$$

$$\Rightarrow 2x + 3y + 4z = 6 \quad \dots(2)$$

It can be seen that the given planes are parallel.

It is known that the distance between two parallel planes, $ax + by + cz = d_1$ and $ax + by + cz = d_2$, is given by,

$$D = \left| \frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}} \right|$$

$$\Rightarrow D = \left| \frac{6 - 4}{\sqrt{(2)^2 + (3)^2 + (4)^2}} \right|$$

$$D = \frac{2}{\sqrt{29}}$$

Thus, the distance between the lines is $\frac{2}{\sqrt{29}}$ units.

Hence, the correct answer is D.

Question 23:

The planes: $2x - y + 4z = 5$ and $5x - 2.5y + 10z = 6$ are

(A) Perpendicular (B) Parallel (C) intersect y-axis

(C) passes through $\left(0, 0, \frac{5}{4}\right)$

Answer

The equations of the planes are

$$2x - y + 4z = 5 \quad \dots (1)$$

$$5x - 2.5y + 10z = 6 \quad \dots (2)$$

It can be seen that,

$$\frac{a_1}{a_2} = \frac{2}{5}$$

$$\frac{b_1}{b_2} = \frac{-1}{-2.5} = \frac{2}{5}$$

$$\frac{c_1}{c_2} = \frac{4}{10} = \frac{2}{5}$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Therefore, the given planes are parallel.

Hence, the correct answer is B.